

LIMITING BEHAVIOR OF A CLASS OF HERMITIAN-YANG-MILLS METRICS

JIXIANG FU

ABSTRACT. We study the limiting behavior of Hermitian-Yang-Mills metrics on a class of rank two slope-stable vector bundles over the product of two elliptic curves with a family of product metrics, which are flat and have areas ϵ and ϵ^{-1} on two factors respectively. The method is to construct a family of Hermitian metrics and then compare them with the normalized Hermitian-Yang-Mills metrics. We find that the metrics are close in C^k to arbitrary order in ϵ .

1. INTRODUCTION

A Calabi-Yau manifold is a compact Kähler manifold with zero first Chern class and vanishing first Betti number. Yau's solution [30] to the Calabi Conjecture provides a unique Ricci-flat Kähler metric in each Kähler class of a Calabi-Yau manifold. Motivated by mirror symmetry and the SYZ conjecture [23], Gross and Wilson [11] initiated the study of the limiting behavior of Yau's Ricci-flat metrics in a large complex structure limit. They considered a general K3 surface that is a hyper-Kähler rotation of an elliptic K3 surface with 24 singular fibers, and showed that its Ricci-flat metrics converge (collapse) to a metric on the base S^2 with singularities on the discriminant locus of 24 points. Later, several other papers (cf. [29, 31, 19, 26, 10]) studied the same question.

In this paper, we will study the Hermitian-Yang-Mills (HYM for brevity) version of the above question. Let V be a slope stable holomorphic vector bundle over a compact Kähler manifold X with a Kähler metric (form) ω . By a theorem of Donaldson [2] and Uhlenbeck-Yau [27], V admits an irreducible HYM metric H , which is unique up to multiplicative constant. Suppose X is a Calabi-Yau manifold with a family of Kähler metrics ω_ϵ approaching a large Kähler metric limit, and suppose V is slope stable with each ω_ϵ , then we obtain a family of HYM metrics H_ϵ .

Question. What is the limiting behavior of H_ϵ , after normalization, when ω_ϵ goes to a large Kähler metric limit?

The Kähler manifold X we consider here is a product $T \times B$ of two copies of the complex one-torus \mathbb{C}/Γ , where $\Gamma = \mathbb{Z} + i\mathbb{Z}$. In this case, a family of product metrics ω_ϵ , which are flat and have areas ϵ and ϵ^{-1} on T and B respectively, approaches a large Kähler metric limit when $\epsilon \rightarrow 0$ (cf. [16]).

The holomorphic vector bundle V over X is constructed as follows (cf. [3, 4]). Let T^* be the dual of T and let $X^* = T^* \times B$. The product $X^* \times_B X$ is a smooth complex threefold. Let Y be a compact (complex) curve of X^* so that the induced projection $\varphi : Y \rightarrow B$ is a two-sheet branched cover. Let

$$\iota : Y \times_B X \longrightarrow X^* \times_B X, \quad p_1 : Y \times_B X \longrightarrow Y, \quad p_2 : Y \times_B X \longrightarrow X$$

be the canonical inclusion and projections. Let \mathcal{P} be the Poincaré line bundle on $X^* \times_B X$. Then for any degree zero line bundle \mathcal{F} over Y , we can form a line bundle $\mathcal{N} = K_Y^{1/2} \otimes \varphi^* K_B^{-1/2} \otimes \mathcal{F}$ over Y and a rank two vector bundle over X

$$V = p_{2*}(\iota^* \mathcal{P} \otimes p_1^* \mathcal{N}).$$

Its first Chern class vanishes. Moreover, by an adiabatic argument (cf. [4]), V is ω_ϵ -slope stable for small ϵ . Therefore there exists a family of irreducible HYM metrics $H_{1,\epsilon}$ on V with respect to ω_ϵ . Because $c_1(V) = 0$, the associated curvatures $\Theta(H_{1,\epsilon})$ satisfy

$$(1.1) \quad \Lambda \Theta(H_{1,\epsilon}) \triangleq \frac{i \Theta(H_{1,\epsilon}) \wedge \omega_\epsilon}{\omega_\epsilon^2} = 0.$$

The purpose of this paper is to investigate the above question for $H_{1,\epsilon}$ when $\epsilon \rightarrow 0$. We construct in section 5 explicitly a family of Hermitian metrics $H_{0,\epsilon}$ on V . Equation (1.1) leads us first to establish

Proposition 1. *For any positive integer l , there is a constant $C = C(l)$ depending only on l and an open cover of X such that for sufficiently small $\epsilon > 0$, the associated curvatures $\Theta(H_{0,\epsilon})$ of $H_{0,\epsilon}$ satisfy*

$$\| \Lambda \Theta(H_{0,\epsilon}) \|_{C^0} \leq C \epsilon^l.$$

We then normalize $H_{1,\epsilon}$ with respect to $H_{0,\epsilon}$ and compare them. The main result of this paper is

Theorem 2. *For any non-negative integer k and positive integer l , there is a constant $C = C(k, l)$ depending on k, l and an open cover of X such that for sufficiently small $\epsilon > 0$,*

$$\| (H_{0,\epsilon})^{-1} H_{1,\epsilon} - \text{Id} \|_{C^k} \leq C \epsilon^l.$$

Here $\Lambda \Theta(H_{0,\epsilon})$ and $(H_{0,\epsilon})^{-1} H_{1,\epsilon}$ are in $\text{End}(V)$ (cf. [2, p.4]), where there is no natural C^k -norm. We use $H_{0,\epsilon}$ to define a C^k -norm. That is, for a local trivialization of V , we choose a unitary frame field relative to $H_{0,\epsilon}$ and define a C^k -norm on $\text{End}(V)$ to be the C^k -norm of the resulting matrix representations. The C^k -norm of a function is defined as in [8, p.53] which does not depend on ϵ . Hence, the metrics $H_{1,\epsilon}$ and $H_{0,\epsilon}$ are close in C^k to arbitrary order in ϵ .

We will prove the above results in the last two sections (see Proposition 10 and Theorem 15). The key step to construct $H_{0,\epsilon}$ is to construct a family of HYM metrics on V over the product of a neighborhood of a branched point in B and the fiber T . In section 3, we construct such metrics (3.8) and hence derive a PDE (4.1) depending on ϵ . This equation has a unique smooth solution and also a singular solution $\frac{1}{2} \ln r$. Moreover, according to Gidas-Ni-Nirenberg's theorem in [7], it can be reduced to an ODE (4.3) on the interval $[0, 2r_0]$, which is a singular perturbed equation, the small parameter is ϵ . We estimate C^k -norm of the difference between the smooth solution and the singular solution on the interval $[r_0, 2r_0]$ in section 4. We find that they are close in C^k to arbitrary order in ϵ . In section 5, we first use the Green function of a degree zero divisor on B to construct a HYM metric on V , which is singular on V over the fiber of every branched point. However, this singular metric is essentially the same as the metrics (3.8) when the PDE (4.1) takes the *singular* solution. Hence, we can glue this metric to the local *smooth* HYM metrics (3.8). The resulting metrics can be normalized conformally to obtain

a family of Hermitian metrics $H_{0,\epsilon}$ so that $\text{Tr } \Lambda \Theta(H_{0,\epsilon}) = 0$. This guarantees that $\det((H_{0,\epsilon})^{-1} H_{1,\epsilon})$ is a constant. Hence, $H_{1,\epsilon}$ can be normalized so that this constant is 1.

We believe that our method can be applied to the case of the elliptic $K3$ surface over \mathbb{P}^1 with a section if we can know its large Kähler metric limit more. Thus, we believe that it may have many important applications to mirror symmetry (cf. [13, 4, 14, 17, 14, 24, 5, 6, 29]).

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2. A LOCALIZATION OF V

In this section, we shall set up the notations that will be followed in this paper. Let $\Gamma = \mathbb{Z} + i\mathbb{Z}$ and Γ^* be the dual of Γ . Let T and B be two copies of the complex one-torus \mathbb{C}/Γ and let $X = T \times B$. Let $T^* = \mathbb{C}^*/\Gamma^*$ be the dual of T and let $X^* = T^* \times B$. Set $z = x_1 + ix_2$, $w = y_1 + iy_2$ and $w^* = y_1^* + iy_2^*$ to be the complex coordinates of B , T and T^* respectively. We endow X with a family of Kähler metrics depending on ϵ :

$$(2.1) \quad \omega_\epsilon = \epsilon dy_1 \wedge dy_2 + \epsilon^{-1} dx_1 \wedge dx_2.$$

By viewing Γ as the transformation group of \mathbb{C} and viewing Γ^* as the transformation group of \mathbb{C}^* , $\mathbb{C}^* \times \mathbb{C}$ becomes the universal cover of $T^* \times T$ with the deck transformation group $\Gamma^* \times \Gamma$:

$$\mathfrak{g}_{(\lambda^*, \lambda)}(w^*, w) = (w^* + \lambda^*, w + \lambda).$$

Hence

$$(\mathbb{C}^* \times \mathbb{C} \times B)/(\Gamma^* \times \Gamma) = X^* \times_B X.$$

We recall the construction of the Poincaré line bundle. Start with the trivial line bundle $\tilde{\mathcal{P}}$ over $\mathbb{C}^* \times T$ with the standard flat connection along \mathbb{C}^* , and with connection which connection form along T at $\{w^*\} \times T$ is

$$(2.2) \quad \theta = -\pi i(w^* d\bar{w} + \overline{w^*} dw).$$

Then we can lift the Γ^* action on \mathbb{C}^* to $\tilde{\mathcal{P}}$, if we denote by $\varepsilon_{(w^*, w)}$ its constant one global section,

$$\mathfrak{g}_{\lambda^*}^* \varepsilon_{(w^* + \lambda^*, w)} = \exp(-\pi i(\lambda^* \bar{w} + \overline{\lambda^*} w)) \varepsilon_{(w^*, w)}.$$

Thus, $\tilde{\mathcal{P}}$ can be reduced to a line bundle \mathcal{P} over $T^* \times T$, which is called the Poincaré line bundle. Moreover, the curvature associated to θ is

$$(2.3) \quad \Theta = -\pi i(dw^* \wedge d\bar{w} + \overline{dw^*} \wedge dw),$$

which is a $(1, 1)$ -form. This makes \mathcal{P} a holomorphic line bundle with holomorphic frame

$$(2.4) \quad \tilde{\varepsilon}_{(w^*, w)} = \exp(\pi i w^* \bar{w}) \varepsilon_{(w^*, w)};$$

it transforms under $\Gamma^* \times \Gamma$ via

$$(2.5) \quad \begin{aligned} \mathfrak{g}_{(0, \lambda)}^* \tilde{\varepsilon}_{(w^*, w + \lambda)} &= \exp(\pi i w^* \bar{\lambda}) \tilde{\varepsilon}_{(w^*, w)}, \\ \mathfrak{g}_{(\lambda^*, 0)}^* \tilde{\varepsilon}_{(w^* + \lambda^*, w)} &= \exp(-\pi i \bar{\lambda}^* w) \tilde{\varepsilon}_{(w^*, w)}. \end{aligned}$$

\mathcal{P} can be viewed as a line bundle over $X^* \times_B X$ by pulling back. By (2.3), we have

$$(2.6) \quad c_1(\mathcal{P}) = \frac{\Theta}{2\pi i} = -\frac{1}{2}(dw^* \wedge d\bar{w} + \overline{dw^*} \wedge dw).$$

As in section 1, we take a (complex) curve Y in X^* such that the restricted map of Y to B is a $2 : 1$ branched cover. Such a curve can be constructed as follows. Pick a curve $Y_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of class

$$[0 \times \mathbb{P}^1] + d[\mathbb{P}^1 \times 0].$$

This can be constructed as the graph of a degree d polynomial $\mathbb{C} \rightarrow \mathbb{C}$. Fix a $2 : 1$ branched cover $T^* \rightarrow \mathbb{P}^1$ and $B \rightarrow \mathbb{P}^1$. We define a curve $Y \subset X^*$ be the pre-image of Y_0 under the natural map $X^* \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Then $\varphi : Y \rightarrow B$ is a $2 : 1$ branched cover and the degree of $q : Y \rightarrow T^*$ is $2d$.

Then as in section 1, we can use Y and \mathcal{P} to construct the rank two vector bundle V over X . From section 7 of [4], we can see that

$$(2.7) \quad \begin{aligned} c_1(V) &= p_{2*}(\iota^* c_1(\mathcal{P})), \\ c_2(V) &= \frac{1}{2}(p_{2*}(\iota^* c_1(\mathcal{P})))^2 - \frac{1}{2}p_{2*}(\iota^* c_1(\mathcal{P})^2), \end{aligned}$$

and $c_1(V) = 0$. (Equation (2.7) can also be derived by curvatures in section 5.) Moreover, by (2.6), we have

$$\begin{aligned} \int_X c_2(V) &= \left(\frac{i}{2}\right)^2 \int_X p_{2*}(dw^* \wedge \overline{dw^*} \wedge dw \wedge \overline{dw}) \\ &= \frac{i}{2} \int_Y q^*(dw^* \wedge \overline{dw^*}) \\ &= \frac{i}{2} \int_{[q_*(Y)]} dw^* \wedge \overline{dw^*} \\ &= \deg(q). \end{aligned}$$

Next we should simplify V . Let

$$D_0 = \sum_{a=1}^n \xi_a$$

be the branched locus on B . By the Riemann-Hurwitz formula, the genus $g(Y)$ of Y is bigger than 1 and $n = 2(g(Y) - 1)$. Since the degree of K_Y is $2(g(Y) - 1)$ and the degree of K_B is 0,

$$\deg(K_Y^{1/2} \otimes \varphi^* K_B^{-1/2}) = g(Y) - 1 = \frac{n}{2}.$$

For simplicity, we assume that $g(Y)$ is odd and hence $4|n$. Then pick a divisor on B

$$D = \sum_{j=n+1}^{\frac{5}{4}n} \xi_j$$

that is disjoint from the branched locus D_0 . Hence

$$\deg(\varphi^*(\mathcal{O}_B(D))) = 2 \deg(\mathcal{O}_B(D)) = \frac{n}{2}.$$

Therefore, \mathcal{N} in the first section can be taken as $\varphi^*(\mathcal{O}_B(D)) \otimes \mathcal{F}'$ for a degree zero line bundle \mathcal{F}' . Without loss of generality, we assume that \mathcal{F}' is trivial. Otherwise, we can tensor a flat metric on \mathcal{F}' with the constructed Hermitian metrics on V in section 5. Thus,

$$V = p_{2*}(\mathcal{L}) \quad \text{for} \quad \mathcal{L} = \iota^* \mathcal{P} \otimes (\varphi \circ p_1)^*(\mathcal{O}_B(D)).$$

For our purpose, we will give a local trivialization of this vector bundle. We denote by d_B the distance on B induced from the Euclidean metric on \mathbb{C} . Hence, d_B does not depend on ϵ . We pick a small $r_0 > 0$ so that the discs

$$U_\alpha = \{z \in B \mid d_B(z, \xi_\alpha) < 2r_0\} \subset B, \quad \alpha = 1, \dots, 5n/4,$$

are disjoint. For such an α , we pick an analytic chart z_α of U_α so that $z_\alpha(\xi_\alpha) = 0$. In the following, for convenience, we will denote $\alpha = 0, 1, \dots, 5n/4$; $a = 1, \dots, n$; and $j = n+1, \dots, 5n/4$.

We first localize \mathcal{L} . Denote $U_0^0 = B \setminus D$. Then $\{U_0^0, U_{n+1}, \dots, U_{\frac{5}{4}n}\}$ is an open cover of B . We can give a local holomorphic frame e_0 of $\mathcal{O}_B(D) \mid_{U_0^0}$ and e_j of $\mathcal{O}_B(D) \mid_{U_j}$ such that over $U_j \cap U_0^0$,

$$e_j = z_j^{-1} e_0.$$

Denote by \mathfrak{U}_j the pre-image of U_j and by \mathfrak{U}_0^0 of U_0^0 in $Y \times_B X$ under the map $\varphi \circ p_1$. Then

$$\tilde{\nu}_j(w^*(z_j), w, z_j) = \tilde{\epsilon}(w^*(z_j), w) \otimes e_j(z_j)$$

forms a local holomorphic frame of $\mathcal{L} \mid_{\mathfrak{U}_j}$ and

$$\tilde{\nu}_0(w^*(z_j), w, z_j) = \tilde{\epsilon}(w^*(z_j), w) \otimes e_0(z_j)$$

of $\mathcal{L} \mid_{\mathfrak{U}_0^0}$. They satisfy, over $\mathfrak{U}_j \cap \mathfrak{U}_0^0$,

$$(2.8) \quad \tilde{\nu}_j(w^*(z_j), w, z_j) = z_j^{-1} \tilde{\nu}_0(w^*(z_j), w, z_j).$$

They also transform under Γ via the first formula in (2.5).

Now we can localize V over X . We take $U_0 = B \setminus (D_0 \cup D_1)$. Then U_0, U_a, U_j also form an open cover of B and their pre-images $\mathcal{U}_0, \mathcal{U}_a, \mathcal{U}_j$ in X form an open cover of X . We have a local holomorphic frame of V over \mathcal{U}_0

$$(2.9) \quad \tilde{\mu}_1^0(w, z) = p_{2*} \tilde{\nu}_0(w_1^*(z), w, z), \quad \tilde{\mu}_2^0(w, z) = p_{2*} \tilde{\nu}_0(w_2^*(z), w, z).$$

Here $w_1^*(z)$ and $w_2^*(z)$ are the two local sections of $\varphi : Y \rightarrow B$ when restricted to U_0 . We caution that the two sections w_1^* and w_2^* only exist locally. But this will not confuse us to construct the Hermitian metrics in section 5. On the other hand, under our assumption, we can assume that $w_1^*(z_j)$ and $w_2^*(z_j)$ are well-defined on U_j . Hence we have a local holomorphic frame of V over \mathcal{U}_j

$$(2.10) \quad \tilde{\mu}_1^j(w, z_j) = p_{2*} \tilde{\nu}_j(w_1^*(z_j), w, z_j), \quad \tilde{\mu}_2^j(w, z_j) = p_{2*} \tilde{\nu}_j(w_2^*(z_j), w, z_j).$$

Combining (2.9) and (2.10) with (2.8) gives the relations over $\mathcal{U}_0 \cap \mathcal{U}_j$

$$(2.11) \quad \tilde{\mu}_1^j = z_j^{-1} \tilde{\mu}_1^0, \quad \tilde{\mu}_2^j = z_j^{-1} \tilde{\mu}_2^0.$$

We next look at \mathcal{U}_a . Since $\varphi : Y \rightarrow B$ is the two-to-one branched cover ramified at ξ_a , we choose w_a^* so that over U_a the curve $Y \subset X^*$ is given by $(w_a^*)^2 = z_a$. Hence the direction image sheaf $\varphi_* \mathcal{O}_Y|_{U_a}$ is a free \mathcal{O}_{U_a} -module generated by 1 and w_a^* . For $V|_{\mathcal{U}_a}$, following (2.9) we can pick $w_1^*(z_a) = \sqrt{z_a}$ and $w_2^*(z_a) = -\sqrt{z_a}$, and set

$$(2.12) \quad \tilde{\mu}_1^a = \frac{1}{\sqrt{2}}(\tilde{\mu}_1^0 + \tilde{\mu}_2^0), \quad \tilde{\mu}_2^a = \frac{\sqrt{z_a}}{\sqrt{2}}(\tilde{\mu}_1^0 - \tilde{\mu}_2^0).$$

The sections $\tilde{\mu}_1^a$ and $\tilde{\mu}_2^a$ are well-defined holomorphic sections of $V|_{\mathcal{U}_a}$ independent of the choice of single-valued branch of $\sqrt{z_a}$; also the two sections $\tilde{\mu}_1^a$ and $\tilde{\mu}_2^a$ generate the holomorphic bundle $V|_{\mathcal{U}_a}$. Thus we can and shall set them to be a holomorphic frame of $V|_{\mathcal{U}_a}$. In other words, (2.12) gives the transition functions over $\mathcal{U}_0 \cap \mathcal{U}_a$ between the frames $(\tilde{\mu}_1^a, \tilde{\mu}_2^a)$ and $(\tilde{\mu}_1^0, \tilde{\mu}_2^0)$.

Similarly, we can also use $\varepsilon_{(w^*, w)}$ to define a smooth frame $(\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha)$ of $V|_{\mathcal{U}_\alpha}$. They also satisfy the relations

$$(2.13) \quad \hat{\mu}_1^j = z_j^{-1} \hat{\mu}_1^0, \quad \hat{\mu}_2^j = z_j^{-1} \hat{\mu}_2^0, \quad \text{over } \mathcal{U}_j \cap \mathcal{U}_0;$$

$$(2.14) \quad \hat{\mu}_1^a = \frac{1}{\sqrt{2}}(\hat{\mu}_1^0 + \hat{\mu}_2^0), \quad \hat{\mu}_2^a = \frac{\sqrt{z_a}}{\sqrt{2}}(\hat{\mu}_1^0 - \hat{\mu}_2^0), \quad \text{over } \mathcal{U}_a \cap \mathcal{U}_0.$$

Finally, by (2.4), the local holomorphic frames are related to the smooth frames:

$$(2.15) \quad (\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha) = (\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha) A_\alpha,$$

where

$$(2.16) \quad A_\alpha = \begin{cases} \begin{pmatrix} \exp(\pi i w_1^*(z) \overline{w}) & 0 \\ 0 & \exp(\pi i w_2^*(z) \overline{w}) \end{pmatrix}, & \alpha = 0, j; \\ \begin{pmatrix} \cosh(\pi i \sqrt{z} \overline{w}) & \sqrt{z} \sinh(\pi i \sqrt{z} \overline{w}) \\ \frac{1}{\sqrt{z}} \sinh(\pi i \sqrt{z} \overline{w}) & \cosh(\pi i \sqrt{z} \overline{w}) \end{pmatrix}, & \alpha = a. \end{cases}$$

3. THE SYSTEM OF HYM CONNECTIONS

In this section we first recall some definitions and notations on connections in Hermitian vector bundles as in Chapter 1 of [12]. (Hence our notations here differs from [9].) Let E be a rank r complex vector bundle over a Kähler manifold (M, ω) . Let D be a connection in E . Let $s_U = (s_1, \dots, s_r)$ be a local frame of E over an open set $U \subset M$. Then we can write

$$Ds_i = \sum s_j \theta_i^j.$$

The matrix 1-form $\theta_U = (\theta_i^j)$ is called the connection form of D with respect to s_U . The curvature form Θ_U of D relative to s_U is defined by

$$\Theta_U = d\theta_U + \theta_U \wedge \theta_U.$$

If $s'_U = (s'_1, \dots, s'_r)$ is another local frame over U , which is related to s_U by

$$(3.1) \quad s_U = s'_U A_U,$$

where $A_U : U \rightarrow GL(r, \mathbb{C})$ is a matrix-valued function on U . Let $\theta'_U = (\theta_j^i)$ and Θ'_U be the connection and curvature form of D relative to s'_U . Then

$$(3.2) \quad \theta_U = A_U^{-1} \theta'_U A_U + A_U^{-1} dA_U,$$

and

$$(3.3) \quad \Theta_U = A_U^{-1} \Theta'_U A_U.$$

Let H be a Hermitian metric on E . We set

$$h_{i\bar{j}} = H(s_i, s_j)$$

and $H_U = (h_{i\bar{j}})$. H_U is a positive definite Hermitian matrix at every point of U . Under a change of frame given by (3.1), we have

$$(3.4) \quad H_U = (A_U)^t H'_U \bar{A}_U.$$

Here $(A_U)^t$ is denoted as the transpose of A_U .

Now if E is a holomorphic vector bundle and H is a Hermitian metric on E , then there exists a canonical connection D_H , which is called the Hermitian connection, defined as follows. Let $\tilde{s}_U = (\tilde{s}_1, \dots, \tilde{s}_r)$ be a local holomorphic frame on U and \tilde{H}_U be the Hermitian matrix for H with respect to \tilde{s}_U . Then the Hermitian connection with respect to \tilde{s}_U is determined by

$$(3.5) \quad (\tilde{\theta}_U)^t = \partial \tilde{H}_U \tilde{H}_U^{-1}$$

and its curvature form is

$$(3.6) \quad \tilde{\Theta}^t = \bar{\partial}(\partial \tilde{H}_U \tilde{H}_U^{-1}),$$

which is a matrix $(1, 1)$ -form. Hence according to (3.3), the curvature form Θ' of the Hermitian connection with respect to any frame s'_U is also a matrix $(1, 1)$ -form. We define

$$(3.7) \quad \Lambda \Theta' = \frac{m \cdot \frac{i}{2} \Theta' \wedge \omega^{m-1}}{\omega^m},$$

where $m = \dim_{\mathbb{C}} M$. If $c_1(V) = 0$, then H is called a HYM metric if and only if

$$\Lambda \Theta' = 0.$$

In the following we shall derive the system of HYM connections of V over \mathcal{U}_a for $1 \leq a \leq n$. Since $V|_{\mathcal{U}_a}$ are essentially the same, we shall work out one of them in detail. For convenience, we shall drop the super(sub)-script a .

We endow $V|_{\mathcal{U}}$ with a class of metrics. Let $u_\epsilon : U \rightarrow \mathbb{R}$ be a real function and set

$$(3.8) \quad \hat{h}_\epsilon = \begin{pmatrix} e^{-u_\epsilon} & 0 \\ 0 & e^{u_\epsilon} \end{pmatrix}.$$

Since u_ϵ does not depend on the variable w , \hat{h}_ϵ gives a Hermitian metric h_ϵ so that it is the Hermitian matrix for h_ϵ in $(\hat{\mu}_1, \hat{\mu}_2)$. Thus, by (3.4) and (2.15),

$$(3.9) \quad \tilde{h}_\epsilon = A^t \hat{h}_\epsilon \bar{A}$$

gives the Hermitian matrix for h_ϵ in $(\tilde{\mu}_1, \tilde{\mu}_2)$, which depends on w . Hence the Hermitian connection also depends on w (see below).

We let D_{h_ϵ} be the Hermitian connection on $(V|_{\mathcal{U}}, h_\epsilon)$; let $\tilde{\theta}_\epsilon$ and $\hat{\theta}_\epsilon$ be the connection forms of D_{h_ϵ} with respect to $(\tilde{\mu}_1, \tilde{\mu}_2)$ and $(\hat{\mu}_1, \hat{\mu}_2)$. Then, by (3.5),

$$(3.10) \quad (\tilde{\theta}_\epsilon)^t = \partial \tilde{h}_\epsilon \cdot \tilde{h}_\epsilon^{-1}$$

and, by (3.2), $\hat{\theta}_\epsilon$ is related to $\tilde{\theta}_\epsilon$ as

$$(3.11) \quad \hat{\theta}_\epsilon = A\tilde{\theta}_\epsilon A^{-1} - dAA^{-1}.$$

Inserting (3.9) into (3.10) and inserting the resulting equation into (3.11), we have

$$\begin{aligned} \hat{\theta}_\epsilon &= -\bar{\partial}AA^{-1} + (\partial\hat{h}_\epsilon \cdot \hat{h}_\epsilon^{-1})^t + (\hat{h}_\epsilon \overline{\partial}AA^{-1}\hat{h}_\epsilon^{-1})^t \\ &= -\pi i \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} d\bar{w} - \pi i \begin{pmatrix} 0 & e^{2u_\epsilon} \\ \bar{z}e^{-2u_\epsilon} & 0 \end{pmatrix} dw + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial u_\epsilon}{\partial z} dz. \end{aligned}$$

Therefore the associated curvature form is

$$\begin{aligned} \hat{\Theta}(h_\epsilon) &= \pi^2(|z|^2 e^{-2u_\epsilon} - e^{2u_\epsilon}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dw \wedge d\bar{w} \\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2 u_\epsilon}{\partial z \partial \bar{z}} dz \wedge d\bar{z} + \text{other terms}, \end{aligned}$$

and thus, by definition (3.7) with $m = 2$ and $\omega = \omega_\epsilon$ in (2.1),

$$\Lambda \hat{\Theta}(h_\epsilon) = \left(\epsilon \frac{\partial^2 u_\epsilon}{\partial z \partial \bar{z}} + \epsilon^{-1} \pi^2(|z|^2 e^{-2u_\epsilon} - e^{2u_\epsilon}) \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Based on this, we see that h_ϵ becomes HYM if u_ϵ satisfies the equation:

$$(3.12) \quad \frac{\partial^2 u_\epsilon}{\partial z \partial \bar{z}} = \pi^2 \epsilon^{-2} (\exp(2u_\epsilon) - |z|^2 \exp(-2u_\epsilon)).$$

4. REDUCTION TO ODE

In this section, we shall study the solution to the Dirichlet problem:

$$(4.1) \quad \begin{cases} \Delta u = 4\pi^2 \epsilon^{-2} (\exp(2u) - r^2 \exp(-2u)) & \text{in } B_{2r_0}(0), \\ u = \frac{1}{2} \ln(2r_0) & \text{on } \partial B_{2r_0}(0), \end{cases}$$

where $x = (x_1, x_2)$ is the standard coordinate of $B_{2r_0}(0)$, $r^2 = x_1^2 + x_2^2$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

Theorem 3. *Equation (4.1) has a unique smooth and radially symmetric solution u_ϵ that satisfies the following estimates:*

(1) *let $v_\epsilon(r) = u_\epsilon(r) - \frac{1}{2} \ln r$, $r \in [r_0, 2r_0]$, then for any positive integer l and k so that $l > k \geq 0$, there is a constant $C = C(r_0, l, k)$ such that for any $0 < \epsilon < 1/8$,*

$$\|v_\epsilon^{(k)}(r)\|_{C^0([r_0, 2r_0])} < C\epsilon^{l-k}.$$

Here $v_\epsilon^{(k)}(r)$ is the k -th derivative of v_ϵ in r ; and

(2) *for any integer $k \geq 0$ and any $R_0 \leq 2r_0$, there exists a constant $C = C(r_0, R, k)$ such that for any $0 < \epsilon < 1/8$,*

$$\|u_\epsilon\|_{C^k(B_R(0))} \leq C\epsilon^{-k-2}.$$

Proof. After substituting \bar{u} for $2u - \ln(2r_0)$, x_1 for $\frac{x_1}{2r_0}$, x_2 for $\frac{x_2}{2r_0}$, r^2 for $\frac{r^2}{4r_0^2}$, and ϵ for $\frac{\epsilon}{4\sqrt{2\pi}}r_0^{-\frac{3}{2}}$, equation (4.1) becomes

$$(4.2) \quad \begin{cases} \Delta \bar{u} = \epsilon^{-2} (\exp \bar{u} - r^2 \exp(-\bar{u})) & \text{in } B_1(0), \\ \bar{u} = 0 & \text{on } \partial B_1(0). \end{cases}$$

The theorem then follows from Proposition 4, Lemma 7 and Proposition 9 below. \square

Proposition 4. *Equation (4.2) has a unique smooth and radially symmetric solution \bar{u}_ϵ that satisfies $\frac{\partial}{\partial r}\bar{u}_\epsilon > 0$ for $0 < r < 1$.*

Proof. Because for each $x = (x_1, x_2)$ the function $\epsilon^{-2}(\exp \bar{u} - r^2 \exp(-\bar{u}))$ is a monotone increasing function of \bar{u} , according to [21] the boundary value problem (4.2) is uniquely solvable.

To prove the second part, we first use the maximum principle to prove that the solution \bar{u}_ϵ to (4.2) is negative. Let $x_0 \in \bar{B}_1(0)$ be such that $\bar{u}_\epsilon(x_0) = \max_{x \in \bar{B}_1(0)} \bar{u}_\epsilon$. In case $\bar{u}_\epsilon(x_0) \geq 0$ and $x_0 \notin \partial B_0$, we have $\exp 2\bar{u}_\epsilon(x_0) - |x_0|^2 > 0$, and that there is a neighborhood $\Omega \subset B_1(0)$ of x_0 such that $\exp 2\bar{u}_\epsilon(x) - |x|^2 > 0$ in Ω . Therefore

$$\Delta \bar{u}_\epsilon = \epsilon^{-2}(\exp \bar{u}_\epsilon - r^2 \exp(-\bar{u}_\epsilon)) > 0, \quad x \in \Omega.$$

Applying the strong maximum principle, we know that the maximum of \bar{u}_ϵ on $\bar{\Omega}$ can be achieved only on $\partial\Omega$, contradicting to that x_0 is a local maximum of \bar{u}_ϵ . This proves that $\bar{u}_\epsilon < 0$ in $B_1(0)$. After this, we can apply Corollary 1 of [7, p.227] to conclude that \bar{u}_ϵ is radially symmetric and $\frac{\partial}{\partial r}\bar{u}_\epsilon > 0$ for all $0 < r < 1$. \square

Because of this, we can reduce (4.2) to an ODE:

$$(4.3) \quad \bar{u}''(r) + r^{-1}\bar{u}'(r) = \epsilon^{-2}(\exp \bar{u}(r) - r^2 \exp(-\bar{u}(r))).$$

Our next goal is to show that the solution $\bar{u}_\epsilon(r)$ is close to $\ln r$ for $r \in [\frac{1}{2}, 1]$ when $\epsilon \rightarrow 0$. We shall set $\bar{v}_\epsilon(r) = \bar{u}_\epsilon(r) - \ln r$ and estimate $\|\bar{v}_\epsilon^{(k)}(r)\|_{C^0([\frac{1}{2}, 1])}$. Clearly, $\bar{v}_\epsilon(1) = 0$ and $\lim_{r \rightarrow 0} \bar{v}_\epsilon(r) = +\infty$.

Lemma 5. *When $0 < r < 1$, $\bar{v}_\epsilon(r)$ satisfies*

$$\bar{v}_\epsilon(r) > 0, \quad \bar{v}_\epsilon'(r) < 0, \quad \bar{v}_\epsilon''(r) > 0 \quad \text{and} \quad \bar{v}_\epsilon'''(r) < 0.$$

Proof. By (4.3), $\bar{v}_\epsilon(r)$ satisfies

$$(4.4) \quad \bar{v}_\epsilon''(r) + r^{-1}\bar{v}_\epsilon'(r) = 2\epsilon^{-2}r \sinh \bar{v}_\epsilon(r).$$

We first use the maximum principle to prove $\bar{v}_\epsilon(r) > 0$. If it is not, let r_0 be the first point in $(0, 1)$ such that $\bar{v}_\epsilon(r_0) = \min_{r \in (0, 1)} \bar{v}_\epsilon(r) \leq 0$. Hence $\bar{v}_\epsilon'(r_0) = 0$, $\bar{v}_\epsilon''(r_0) \geq 0$. Therefore (4.4) implies $\bar{v}_\epsilon(r_0) = 0$. Then we can assume that there exists a $r_1 \in (r_0, 1)$ such that $\bar{v}_\epsilon(r_1) = \max_{r \in (r_0, 1)} \bar{v}_\epsilon(r) > 0$. Thus, $\bar{v}_\epsilon'(r_1) = 0$, $\bar{v}_\epsilon''(r_1) \leq 0$. This contradicts to (4.4). Hence $\bar{v}_\epsilon(r) > 0$ for all $0 < r < 1$.

Now applying Theorem 3 in [7] to equation (4.4) gives $\bar{v}_\epsilon'(r) < 0$ for $r \in [\frac{1}{2}, 1]$. We claim that this inequality holds for all $r \in (0, 1)$. Otherwise, there exists a $r_2 \in (0, \frac{1}{2})$ such that $\bar{v}_\epsilon'(r_2) = 0$ and $\bar{v}_\epsilon'(r) < 0$ for any $r > r_2$. Hence, $\bar{v}_\epsilon''(r_2) \leq 0$ and therefore (4.4) implies $\sinh \bar{v}_\epsilon(r_2) \leq 0$ or $\bar{v}_\epsilon(r_2) \leq 0$. It is a contradiction.

The inequality for the second derivative follows directly from (4.4). Differentiating (4.4) with respect to r and using (4.4) again, we have

$$(4.5) \quad \bar{v}_\epsilon'''(r) = 2(r^{-2} + \epsilon^{-2}r \cosh \bar{v}_\epsilon(r)) \bar{v}_\epsilon'(r).$$

Hence $\bar{v}_\epsilon'''(r) < 0$ follows. \square

For $t \in (0, 1]$, we set

$$M_i(t) = \begin{cases} \max_{r \in [t, 1]} |\bar{v}_\epsilon^{(i)}(r)|, & \text{for } i = 0, 1, 2; \\ \max_{r \in [t, 1]} |\sinh \bar{v}_\epsilon(r)|, & \text{for } i = 3. \end{cases}$$

Furthermore, $M_i(t)$ is strictly decreasing in $t \in (0, 1)$; and $M_0(t) < M_3(t)$.

We need the inequality

$$(4.6) \quad M_3(1/4) \leq 2^8 \epsilon^2.$$

Rewrite (4.3):

$$(r\overline{u}'_\epsilon(r))' = 2\epsilon^{-2}r^2 \sinh \overline{v}_\epsilon(r);$$

and integrate over $[0, 1]$:

$$\overline{u}'_{\epsilon-}(1) = \int_0^1 (r\overline{v}'_\epsilon(r))' dr = \int_0^1 2\epsilon^{-2}r^2 \sinh \overline{v}_\epsilon(r) dr.$$

On the other hand, Lemma 5 implies

$$\overline{u}'_{\epsilon-}(1) = \lim_{r \rightarrow 1-0} \frac{\overline{u}_\epsilon(r) - \overline{u}_\epsilon(1)}{r - 1} \leq \lim_{r \rightarrow 1-0} \frac{\ln r - \ln 1}{r - 1} = 1.$$

Hence,

$$(4.7) \quad \int_0^1 r^2 \sinh \overline{v}_\epsilon(r) dr \leq \epsilon^2/2.$$

As $\sinh \overline{v}_\epsilon(r)$ is strictly decreasing,

$$(1/8)^2 \sinh \overline{v}_\epsilon(1/4) < r^2 \sinh \overline{v}_\epsilon(r) \quad \text{for } r \in [1/8, 1/4].$$

Integrating over $[1/8, 1/4]$ and using (4.7), we obtain

$$(1/8)^3 \sinh \overline{v}_\epsilon(1/4) < \epsilon^2/2.$$

This proves (4.6).

We need more estimates on $M_i(t)$.

Lemma 6. *For any $t, t' \in [1/4, 1/2]$ and for any $0 < \epsilon < 1/8$, we have*

- (1) $M_2(t) = \frac{2t}{\epsilon^2} M_3(t) + \frac{1}{t} M_1(t)$;
- (2) $M_1(t) < \frac{2}{t} M_3(t)$; and
- (3) $M_3(t') < \frac{2}{t'-t} \epsilon^2 M_1(t)$, for $t' > t$.

Proof. The first follows directly from (4.4) and Lemma 5. We now prove (2). For $1/4 \leq t \leq 1/2$ and $0 < \epsilon < 1/8$, the Taylor expansion of $\overline{v}_\epsilon(r)$ at $r = t$ is

$$\overline{v}_\epsilon(t + \epsilon) = \overline{v}_\epsilon(t) + \overline{v}'_\epsilon(t)\epsilon + \overline{v}''_\epsilon(t + \eta\epsilon)\epsilon^2/2, \quad 0 \leq \eta \leq 1.$$

Then, by Lemma 5, we estimate

$$0 > \overline{v}'_\epsilon(t)\epsilon = \overline{v}_\epsilon(t + \epsilon) - \overline{v}_\epsilon(t) - \overline{v}''_\epsilon(t + \eta\epsilon)\epsilon^2/2 > -\overline{v}_\epsilon(t) - \overline{v}''_\epsilon(t)\epsilon^2/2.$$

Hence,

$$M_1(t) < \epsilon^{-1} M_0(t) + (\epsilon/2) M_2(t) < \epsilon^{-1} M_3(t) + (\epsilon/2) M_2(t).$$

Substituting (1) into the above inequality, we obtain

$$M_1(t) < \epsilon^{-1} M_3(t) + t\epsilon^{-1} M_3(t) + (\epsilon/2)t M_1(t),$$

and therefore,

$$M_1(t) < \frac{1+t}{\epsilon(1-\frac{\epsilon}{2t})} M_3(t) \leq \frac{2}{\epsilon} M_3(t).$$

This proves (2).

For (3), we can rewrite (4.4) as

$$(r\overline{v}'_\epsilon(r))' = 2\epsilon^{-2}r^2 \sinh \overline{v}_\epsilon(r).$$

Integrating over $[t, 1]$ and using Lemma 5, we get

$$(4.8) \quad 2\epsilon^{-2} \int_t^1 r^2 \sinh \bar{v}_\epsilon(r) dr = \bar{v}'_{\epsilon-}(1) - t\bar{v}'_\epsilon(t) \leq t|\bar{v}_\epsilon(t)| = tM_1(t).$$

On the other hand, as in the proof of inequality (4.6), we have

$$(4.9) \quad \begin{aligned} 2\epsilon^{-2} \int_t^1 r^2 \sinh \bar{v}_\epsilon(r) dr &\geq 2\epsilon^{-2} \int_t^{t'} r^2 \sinh \bar{v}_\epsilon(r) dr \\ &\geq 2\epsilon^{-2} t^2 (t' - t) \sinh \bar{v}_\epsilon(t') \geq 2\epsilon^{-2} t^2 (t' - t) M_3(t'). \end{aligned}$$

Combining (4.8) with (4.9) gives (3). \square

We are ready to prove

Lemma 7. *For any positive integer l and non-negative integer k so that $l > k \geq 0$, there exists a constant $C = C(l, k)$ such that for $0 < \epsilon < 1/8$,*

$$\|\bar{v}_\epsilon^{(k)}(r)\|_{C^0([\frac{1}{2}, 1])} \leq C\epsilon^{l-k}.$$

Proof. According to our definitions, $\|\bar{v}_\epsilon^{(k)}(r)\|_{C^0([\frac{1}{2}, 1])} = M_k(\frac{1}{2})$ for $k = 0, 1, 2$. We first look at the case for $k = 0$. By Lemma 6, we have

$$M_3(t') \leq \frac{2^2}{t' - t} \epsilon M_3(t), \quad \text{for } t' > t.$$

Then by the iterated method and by (4.6), we get

$$\begin{aligned} M_3\left(\frac{1}{2} \cdot \frac{l-1}{l}\right) &\leq 2^3 l(l-1) \epsilon M_3\left(\frac{1}{2} \cdot \frac{l-2}{l-1}\right) \\ &\leq (2^3)^{l-2} l(l-1)^2 \dots 3^2 \cdot 2\epsilon^{l-2} M_3(1/4) \\ &\leq 2^{3l+1} (l!)^2 l^{-1} \epsilon^l. \end{aligned}$$

Hence

$$M_0(1/2) \leq M_3(1/2) \leq M_3\left(\frac{1}{2} \cdot \frac{l-1}{l}\right) \leq 2^{3l+1} (l!)^2 l^{-1} \epsilon^l.$$

This proves the case for $k = 0$.

The case for $k = 1$ follows from Lemma 6:

$$M_1(1/2) < M_1\left(\frac{1}{2} \cdot \frac{l-1}{l}\right) \leq \frac{2}{\epsilon} M_3\left(\frac{1}{2} \cdot \frac{l-1}{l}\right) \leq 2^{3l+2} (l!)^2 l^{-1} \epsilon^{l-1}.$$

The case for $k = 2$ follows from the first two cases and Lemma 6.

For the case $k \geq 3$, taking the derivatives to two sides of (4.5) and using the inductive method gives the result. \square

Now we estimate $\|\bar{u}_\epsilon\|_{C^{k,\delta}(B_1(0))}$, which will be used in the last section. In this time, for brevity set

$$F(\bar{u}_\epsilon, r) = \epsilon^{-2} (\exp \bar{u}_\epsilon - r^2 \exp(-\bar{u}_\epsilon)).$$

We denote by F_1 and F_2 the derivatives of F in the first variable and the second variable respectively; we also have the notations F_{11} , F_{12} , F_{22} and so on. For example,

$$\begin{aligned} F_1 &= \epsilon^{-2} (\exp \bar{u}_\epsilon + r^2 \exp(-\bar{u}_\epsilon)), \quad F_2 = -2r\epsilon^{-2} \exp(-\bar{u}_\epsilon); \\ F_{11} &= F, \quad F_{12} = -F_2, \quad F_{22} = -2\epsilon^{-2} \exp(-\bar{u}_\epsilon) = r^{-1} F_2; \end{aligned}$$

and

$$F_{221} = -F_{22}, \quad F_{222} = 0.$$

Lemma 8. *For any $0 < r < 1$,*

(1) $0 < F < \frac{1}{\epsilon^2}$, $0 < F_1 < \frac{2}{\epsilon^2}$, $-\frac{2}{\epsilon^2} < F_2 < 0$; and

(2) $\|\bar{u}_\epsilon\|_{C^0} \leq \frac{1}{2\epsilon}$, $\|\bar{u}'_\epsilon\|_{C^0} \leq \frac{1}{2\epsilon}$, $\|\bar{u}''_\epsilon\|_{C^0} \leq \frac{3}{2\epsilon^2}$.

Proof. By Proposition 4 and Lemma 5,

$$(4.10) \quad \ln r < \bar{u}_\epsilon(r) < 0 \quad \text{and} \quad 0 < \bar{u}'_\epsilon(r) < r^{-1}.$$

Hence the first two items in (1) are valid. The third item in (1) is from the derivative of F_2 in r

$$F'_2 = -2\epsilon^{-2}(1 - r\bar{u}'_\epsilon(r)) \exp(-\bar{u}_\epsilon) < 0.$$

As to the items in (2), we consider the inequality

$$0 < (r\bar{u}'_\epsilon(r))' = rF < r\epsilon^{-2}.$$

Integration by parts gives

$$(4.11) \quad 0 < \bar{u}'_\epsilon(r) < (r/2)\epsilon^{-2}.$$

Combined with the second inequality in (4.10), we see that when $r > \epsilon$, $0 < \bar{u}'_\epsilon(r) < \frac{1}{\epsilon}$ and when $r \leq \epsilon$, $0 < \bar{u}'_\epsilon(r) < \frac{1}{2\epsilon}$. Hence, we get the second inequality in (2). The first item in (2) is from

$$0 < -\bar{u}_\epsilon(0) = \bar{u}_\epsilon(1) - \bar{u}_\epsilon(0) = \int_0^1 \bar{u}'_\epsilon(r) dr \leq (2\epsilon)^{-1}.$$

Now by (4.3), the third inequality in (2) is direct from the first inequality in (1) and (4.11). \square

Proposition 9. *For any ball $B_R(0) \subset B_1(0)$ and any non-negative integer k , there exists a constant $C = C(R, k)$ such that for any positive ϵ small enough,*

$$\|\bar{u}_\epsilon\|_{C^k(B_R(0))} \leq C\epsilon^{-k-2}.$$

Proof. We need to prove that for any $k \geq 3$,

$$(4.12) \quad \|\nabla^k \bar{u}_\epsilon\|_{L^2(B_R(0))} \leq C\epsilon^{-k}.$$

We assume that \bar{u}_ϵ has the compact support in $B_1(0)$ and $B_R(0) = B_1(0)$; otherwise one can use cut-off functions. The estimates for $k = 3, 4$ are obvious and we omit the proof. We first estimate $\|\nabla^5 \bar{u}_\epsilon\|_{L^2(B_1(0))}$.

By a direct calculation, we have

$$\Delta^2 \bar{u}_\epsilon = \Delta F = F(F_1 + \bar{u}_\epsilon'^2) + 2F_{22}(1 - r\bar{u}_\epsilon')$$

and

$$\begin{aligned} |\nabla \Delta^2 \bar{u}_\epsilon|^2 &= |(F_1 \bar{u}_\epsilon' + F_2)(F_1 + \bar{u}_\epsilon'^2) + F(F \bar{u}_\epsilon' + F_{12} + 2\bar{u}_\epsilon' \bar{u}_\epsilon'') \\ &\quad - 2F_{22} \bar{u}_\epsilon'(1 - r\bar{u}_\epsilon') - 2rF F_{22}|^2. \end{aligned}$$

Then the above lemma implies

$$|\nabla \Delta^2 \bar{u}_\epsilon|^2 \leq C\epsilon^{-10} + C\epsilon^{-6} \exp(-2\bar{u}_\epsilon)$$

for a generic constant C . But integration by parts and the above lemma yields

$$\begin{aligned} \int_{B_1(0)} \exp(-2\bar{u}_\epsilon) dx_1 dx_2 &= 2\pi \int_0^1 r \exp(-2\bar{u}_\epsilon) dr \\ &= \pi r^2 \exp(-2\bar{u}_\epsilon)|_0^1 + 2\pi \int_0^1 r^2 \exp(-2\bar{u}_\epsilon) \bar{u}'_\epsilon dr \leq \pi + C\epsilon^{-1}. \end{aligned}$$

Thus,

$$\| \nabla^5 \bar{u}_\epsilon \|_{L^2(B_1(0))}^2 \leq - \int_{B_1(0)} u \Delta^5 u = \int_{B_1(0)} | \nabla \Delta^2 \bar{u}_\epsilon |^2 \leq C\epsilon^{-10}.$$

In this way, we can prove (4.12) for any $k \geq 6$. The only trouble is to estimate $\int_0^1 r \exp(-2p\bar{u}_\epsilon) dr$ for any positive integer p . But this can be done by using integration by parts m times.

Combined with Lemma (8), we get

$$\| \bar{u}_\epsilon \|_{W^{k,2}(B_1(0))} \leq C\epsilon^{-k}.$$

The sobolev inequality [8, P.171] then gives, for any $0 < \delta < 1$

$$(4.13) \quad \| \bar{u}_\epsilon \|_{C^{k,\delta}(B_1(0))} \leq C\epsilon^{-(k+2)}.$$

This prove the proposition. \square

5. CONSTRUCTION OF A FAMILY OF HERMITIAN METRICS

In this section, if H is a Hermitian metric on V , we will denote by D_H the associated Hermitian connection, by $\hat{\Theta}(H)$ and $\tilde{\Theta}(H)$ the curvature forms of D_H relative to the smooth frames $(\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha)$ and the holomorphic frames $(\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha)$.

Following the convention in section 2, ξ_a is a branched point on B and ξ_j is a point in the support of D . Let

$$\tilde{D} = \sum \xi_a - 4 \sum \xi_j$$

be a new divisor on B which degree is zero. Let G be the Green function of \tilde{D} [16, p.339-340] whose local expansion near ξ_α for $1 \leq \alpha \leq 5n/4$ has the form

$$G(z_\alpha) = -c_\alpha \log |z| + 2g_\alpha(z_\alpha)$$

for the constant $c_a = 1$ or $c_j = -4$ and some harmonic function g_α . We assume that r_0 is small enough so that $G|_{U_\alpha}$ has the above local expansion.

We now construct a Hermitian metric on V using the Green function G and the HYM metric h_ϵ^a , which is denoted as h_ϵ in section 3. Over \mathcal{U}_0 , we define h_0 to be the metric given by the Hermitian matrix valued function for $(\hat{\mu}_1^0, \hat{\mu}_2^0)$

$$\hat{h}_0 = e^{\frac{1}{2}G} I,$$

where I is the 2×2 identity matrix. Thus, the ambiguity of choosing $(\hat{\mu}_1^0, \hat{\mu}_2^0)$ in section 2 is irrelevant. By (3.4) and the notation in (2.15), the Hermitian matrix for h_0 in $(\tilde{\mu}_1^0, \tilde{\mu}_2^0)$ is

$$\tilde{h}_0 = (A_0)^t \hat{h}_0 \overline{A}_0.$$

Since G is harmonic, direct calculation as in section 3 gives

$$(5.1) \quad \begin{aligned} \tilde{\Theta}(h_0) = \hat{\Theta}(h_0) = & -\pi i \begin{pmatrix} \frac{\partial w_1^*(z)}{\partial z} & 0 \\ 0 & \frac{\partial w_2^*(z)}{\partial z} \end{pmatrix} dz \wedge d\bar{w} \\ & + \pi i \begin{pmatrix} \frac{\partial \bar{w}_1^*(z)}{\partial \bar{z}} & 0 \\ 0 & \frac{\partial \bar{w}_2^*(z)}{\partial \bar{z}} \end{pmatrix} dw \wedge d\bar{z}. \end{aligned}$$

Hence, h_0 is a HYM metric on $V|_{\mathcal{U}_0}$. For $n+1 \leq j \leq 5n/4$, because of (2.13), the metric h_0 under $(\hat{\mu}_1^j, \hat{\mu}_2^j)$ over $\mathcal{U}_j \cap \mathcal{U}_0$ is given by the matrix valued function

$$\hat{h}_j = e^{g_j} I.$$

In this way h_0 extends to a smooth metric over \mathcal{U}_j . But over \mathcal{U}_a 's, because of (2.14), the metric h_0 under $(\hat{\mu}_1^a, \hat{\mu}_2^a)$ has the form

$$\hat{h}_a = e^{g_a} \begin{pmatrix} |z|^{-\frac{1}{2}} & 0 \\ 0 & |z|^{\frac{1}{2}} \end{pmatrix}.$$

Clearly, h_0 does not extend to the point ξ_a . However, in section 3 we have found a new HYM metric h_ϵ^a of $V|_{\mathcal{U}_a}$ that under the frame $(\hat{\mu}_1^a, \hat{\mu}_2^a)$ has the form

$$\hat{h}_\epsilon^a = \begin{pmatrix} e^{-u_\epsilon} & 0 \\ 0 & e^{u_\epsilon} \end{pmatrix}$$

of which u_ϵ is the solution to the equation (4.1). We let $h_{a,\epsilon} = e^{g_a} h_\epsilon^a$; then $h_{a,\epsilon}$ is also a HYM metric on $V|_{\mathcal{U}_a}$.

What we shall do is to interpolate the two metrics h_0 and $h_{a,\epsilon}$ over \mathcal{U}_a . We let

$$\rho : (0, (2r_0)^2) \rightarrow [0, 1]$$

be a fixed C^∞ cut-off function with $\rho(r^2) = 1$ for $r < r_0$, $\rho(r^2) = 0$ for $r \geq \frac{4}{3}r_0$. We then define

$$\mathbf{h}_\epsilon|_{\mathcal{U}_a} = (1 - \rho(|z|^2))h_0 + \rho(|z|^2)h_{a,\epsilon}.$$

It is a smooth Hermitian metric on $V|_{\mathcal{U}_a}$ that coincides with h_0 for $|z| \geq \frac{4}{3}r_0$ and coincides with $h_{a,\epsilon}$ for $|z| \leq r_0$. After working this out for all branched points, we obtain a global Hermitian metric \mathbf{h}_ϵ that is h_0 on $V|_{X - \cup_{a=1}^r \mathcal{U}_a(\frac{4}{3}r_0)}$ and $h_{a,\epsilon}$ on $V|_{\mathcal{U}_a(r_0)}$. Here we denote by $\mathcal{U}_a(r)$ the pre-image in X of $U_a(r)$, which is the disc in B with center ξ_a and radius r . From now on we denote $U_0 = B - (D \cup (\cup_{a=1}^n U_a(\frac{3}{2}r_0)))$, $\mathcal{U}_0 = U_0 \times T^2$, and take the corresponding trivialization of V .

Hence, over \mathcal{U}_0 and \mathcal{U}_j , $\hat{\Theta}(\mathbf{h}_\epsilon) = \hat{\Theta}(h_0)$. Over \mathcal{U}_a , direct calculation gives

$$(5.2) \quad \begin{aligned} \hat{\Theta}(\mathbf{h}_\epsilon) = & \pi^2(|z|^2 \exp(\phi_1 - \phi_2) - \exp(\phi_2 - \phi_1)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dw \wedge d\bar{w} \\ & - \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial z \partial \bar{z}} & 0 \\ 0 & \frac{\partial^2 \phi_2}{\partial z \partial \bar{z}} \end{pmatrix} dz \wedge d\bar{z} - \pi i \begin{pmatrix} 0 & 1 + z \frac{\partial(\phi_1 - \phi_2)}{\partial z} \\ \frac{\partial(\phi_2 - \phi_1)}{\partial z} & 0 \end{pmatrix} dz \wedge d\bar{w} \\ & - \pi i \begin{pmatrix} 0 & \exp(\phi_2 - \phi_1) \frac{\partial(\phi_2 - \phi_1)}{\partial \bar{z}} \\ \exp(\phi_1 - \phi_2)(1 + \bar{z} \frac{\partial(\phi_1 - \phi_2)}{\partial \bar{z}}) & 0 \end{pmatrix} d\bar{z} \wedge dw. \end{aligned}$$

where

$$\phi_1 = \ln((1 - \rho)r^{-\frac{1}{2}} + \rho \exp(-u_\epsilon)) \quad \text{and} \quad \phi_2 = \ln((1 - \rho)r^{\frac{1}{2}} + \rho \exp(u_\epsilon)).$$

Notice that near the boundary of \mathcal{U}_a the functions ϕ_1 and ϕ_2 reduces to $-\frac{1}{2}\ln r$ and $\frac{1}{2}\ln r$, and their sum $\phi_1 + \phi_2$ vanishes. Hence we can extend $\phi_1 + \phi_2$ to all X by assigning zero to it away from all \mathcal{U}_a . This point will be used in the following normalization.

Now, over $\hat{X} - \cup_1^n \mathcal{U}_a$, by (5.1) and the notations in section 2, we have

$$(5.3) \quad \text{Tr} \hat{\Theta}(\mathbf{h}_\epsilon) = -\pi i p_{2*}(dw^* \wedge \bar{d}w + \bar{d}w^* \wedge dw)$$

and

$$(5.4) \quad \text{Tr}(\hat{\Theta}(\mathbf{h}_\epsilon) \wedge \hat{\Theta}(\mathbf{h}_\epsilon)) = -2\pi^2 p_{2*}(dw^* \wedge \bar{d}w^* \wedge dw \wedge \bar{d}w).$$

Over \mathcal{U}_a for $1 \leq a \leq n$, by (5.2), we have

$$(5.5) \quad \text{Tr}(\hat{\Theta}(\mathbf{h}_\epsilon)) = -\partial \bar{\partial}(\phi_1 + \phi_2)$$

and

$$\text{Tr}(\hat{\Theta}(\mathbf{h}_\epsilon) \wedge \hat{\Theta}(\mathbf{h}_\epsilon)) = -2\pi^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} dz \wedge d\bar{z} \wedge dw \wedge \bar{d}w,$$

where $\varphi = |z|^2 \exp(\phi_1 - \phi_2) + \exp(\phi_2 - \phi_1)$. Moreover, when $r_0 \rightarrow 0$,

$$(5.6) \quad \int_{\mathcal{U}_a} \text{Tr}(\hat{\Theta}(\mathbf{h}_\epsilon) \wedge \hat{\Theta}(\mathbf{h}_\epsilon)) = 8\pi^2 \int_{\mathcal{U}_a} (\varphi''(r) + \frac{1}{r}\varphi'(r)) r dr d\theta = 16\pi^2 r_0 \rightarrow 0.$$

Combining (5.3) with (5.5) and combining (5.4) with (5.6), we get (2.7) by (2.6).

We need to modify the metric \mathbf{h}_ϵ conformally. From (5.2) we have

$$\text{Tr}(\Lambda \hat{\Theta}(\mathbf{h}_\epsilon)) = -\epsilon \frac{\partial^2(\phi_1 + \phi_2)}{\partial z \partial \bar{z}}.$$

To make it vanish, we will normalize \mathbf{h}_ϵ conformally by the factor $\exp(-\frac{1}{2}(\phi_1 + \phi_2))$:

$$H_{0,\epsilon} = \exp(-\frac{1}{2}(\phi_1 + \phi_2)) \cdot \mathbf{h}_\epsilon.$$

Consequently,

$$(5.7) \quad \text{Tr}(\Lambda \hat{\Theta}(H_{0,\epsilon})) = 0.$$

Moreover, by our construction, $\Lambda \hat{\Theta}(H_{0,\epsilon}) = 0$ over \mathcal{U}_0 , \mathcal{U}_j and $\mathcal{U}_a(r_0)$; and

$$(5.8) \quad \Lambda \hat{\Theta}(H_{0,\epsilon}) = \psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

over $\mathcal{U}_a \setminus \mathcal{U}_a(r_0)$ for the function

$$(5.9) \quad \psi = \frac{1}{\epsilon} \pi^2 (|z|^2 \exp(\phi_1 - \phi_2) - \exp(\phi_2 - \phi_1)) - \frac{\epsilon}{2} \frac{\partial^2(\phi_1 - \phi_2)}{\partial z \partial \bar{z}}.$$

On the other hand, by the first part of Theorem 3, the function ψ satisfies, for any l and $0 < \epsilon < 1/8$,

$$(5.10) \quad \|\psi\|_{C^0([r_0, 2r_0])} \leq C(l, r_0) \epsilon^{l-1}.$$

Therefore we have the following desired estimates immediately

$$(5.11) \quad \|\Lambda \hat{\Theta}(H_{0,\epsilon})\|_{C^0(\mathcal{U}_\alpha)} \leq C(l, r_0) \epsilon^{l-1}.$$

If we use $(\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha)$, since by (3.3) $\tilde{\Theta}(H_{0,\epsilon}) = A_\alpha^{-1} \hat{\Theta}(H_{0,\epsilon}) A_\alpha$ and A_α is fixed which does not depend on ϵ , we also have,

$$(5.12) \quad \|\Lambda \tilde{\Theta}(H_{0,\epsilon})\|_{C^0(\mathcal{U}_\alpha)} \leq C(l, r_0) \epsilon^{l-1}.$$

Finally, by our construction, $(\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha)$ is orthogonal for $H_{0,\epsilon}$. It can be normalized to a unitary frame $(\check{\mu}_1^\alpha, \check{\mu}_2^\alpha)$:

$$(5.13) \quad (\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha) = (\check{\mu}_1^\alpha, \check{\mu}_2^\alpha) N_\alpha$$

where

$$(5.14) \quad N_\alpha = \begin{cases} e^{\frac{1}{4}G} I, & \text{when } \alpha = 0, \\ e^{\frac{1}{2}g_\alpha} I, & \text{when } \alpha = j, \\ e^{\frac{1}{2}g_\alpha} \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix} & \text{when } \alpha = a; \end{cases}$$

where

$$(5.15) \quad \kappa = \left(\frac{((1-\rho)r^{+\frac{1}{2}} + \rho e^{+u_\epsilon})}{((1-\rho)r^{-\frac{1}{2}} + \rho e^{-u_\epsilon})} \right)^{\frac{1}{4}}.$$

Combining (5.13) with (2.15), we have

$$(5.16) \quad (\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha) = (\check{\mu}_1^\alpha, \check{\mu}_2^\alpha) B_\alpha, \quad B_\alpha = N_\alpha A_\alpha.$$

If we denote by $\check{\Theta}(H_{0,\epsilon})$ the curvature form of $D_{H_{0,\epsilon}}$ relative to $(\check{\mu}_1^\alpha, \check{\mu}_2^\alpha)$, then (3.3) gives

$$(5.17) \quad \check{\Theta}(H_{0,\epsilon}) = N_\alpha \hat{\Theta}(H_{0,\epsilon}) N_\alpha^{-1}.$$

Hence, N_α and $\Lambda \hat{\Theta}(H_{0,\epsilon})$ being diagonal makes

$$(5.18) \quad \Lambda \check{\Theta}(H_{0,\epsilon}) = \Lambda \hat{\Theta}(H_{0,\epsilon}).$$

Thus, by (5.11), we get Proposition 1:

Proposition 10. *For any positive integer l , there is a constant $C = C(l, r_0)$ such that for any $0 < \epsilon < 1/8$,*

$$\| \Lambda \check{\Theta}(H_\epsilon^0) \|_{C^0(\mathcal{U}_\alpha)} < C \epsilon^{l-1}.$$

6. LIMITING BEHAVIOR OF HYM METRICS

In this section, when we are working with a single frame, we often drop the superscript and subscript α .

Because V is stable, following the work of Donaldson [2] and of Uhlenbeck-Yau [27], V admits a HYM metric $H_{1,\epsilon}$, which is unique up to scale, with respect to the Kähler metric ω_ϵ . For this metric, denote by $\tilde{H}_{1,\epsilon}$ and $\check{H}_{1,\epsilon}$ the Hermitian matrices relative to $(\tilde{\mu}_1, \tilde{\mu}_2)$ and $(\check{\mu}_1, \check{\mu}_2)$. We will compare $H_{1,\epsilon}$ with $H_{0,\epsilon}$. The method is to estimate $\| \tilde{H}_{1,\epsilon} - I \|_{C^k(\mathcal{U}_\alpha)}$.

As $H_{1,\epsilon}$ and $H_{0,\epsilon}$ are Hermitian metrics on V , there exists an element $H_\epsilon \in \text{End}(V)$ such that

$$H_{0,\epsilon}(H_\epsilon \cdot, \cdot) = H_{1,\epsilon}(\cdot, \cdot).$$

H_ϵ is denoted by $(H_{0,\epsilon})^{-1} H_{1,\epsilon}$ in section 1 (cf. [2, p.4]). We write $H_\epsilon(\tilde{\mu}_1, \tilde{\mu}_2) = (\tilde{\mu}_1, \tilde{\mu}_2) \tilde{H}_\epsilon$ and $H_\epsilon(\check{\mu}_1, \check{\mu}_2) = (\check{\mu}_1, \check{\mu}_2) \check{H}_\epsilon$. Consequently,

$$(6.1) \quad \tilde{H}_{1,\epsilon} = (\tilde{H}_\epsilon)^t \quad \text{and} \quad \check{H}_{1,\epsilon} = (\check{H}_\epsilon)^t \cdot \tilde{H}_{0,\epsilon}.$$

We also have

$$(6.2) \quad \tilde{H}_{1,\epsilon} = B^t (\check{H}_\epsilon)^t \bar{B},$$

which will be used in the estimates for higher order derivatives. By (5.16), we prove this equality as follows:

$$\begin{aligned}
 (\tilde{H}_{1,\epsilon})_{i\bar{j}} &= H_{1,\epsilon}(\tilde{\mu}_i, \tilde{\mu}_j) = H_{0,\epsilon}(H_\epsilon(\tilde{\mu}_i), \tilde{\mu}_j) \\
 &= H_{0,\epsilon}(H_\epsilon(b_{ki}\tilde{\mu}_k), b_{lj}\tilde{\mu}_l) \\
 &= b_{ki}\bar{b}_{lj}H_{0,\epsilon}((\check{H}_\epsilon)_{mk}\tilde{\mu}_m, \tilde{\mu}_l) \\
 &= b_{ki}\bar{b}_{lj}(\check{H}_\epsilon)_{lk}.
 \end{aligned}$$

As $H_{1,\epsilon}$ is the HYM metric, by (3.6) and the second equality in (6.1), direct computation as in [27, p.S264] yields

$$\begin{aligned}
 (6.3) \quad 0 &= \Lambda\tilde{\Theta}(H_{1,\epsilon}) = \Lambda\bar{\partial}(\partial\tilde{H}_{1,\epsilon} \cdot (\tilde{H}_{1,\epsilon})^{-1})^t \\
 &= \Lambda\bar{\partial}(\tilde{H}_\epsilon^{-1}\partial\tilde{H}_\epsilon) + \Lambda\tilde{H}_\epsilon^{-1}\tilde{\Theta}(H_{0,\epsilon})\tilde{H}_\epsilon \\
 &\quad - \Lambda\tilde{H}_\epsilon^{-1} \cdot \bar{\partial}\tilde{H}_\epsilon \cdot \tilde{H}_\epsilon^{-1} \wedge (\partial\tilde{H}_{0,\epsilon} \cdot (\tilde{H}_{0,\epsilon})^{-1})^t \cdot \tilde{H}_\epsilon \\
 &\quad - \Lambda\tilde{H}_\epsilon^{-1} \cdot (\partial\tilde{H}_{0,\epsilon} \cdot (\tilde{H}_{0,\epsilon})^{-1})^t \wedge \bar{\partial}\tilde{H}_\epsilon.
 \end{aligned}$$

Taking the trace of the above system and combining with $\text{Tr}(\Lambda\tilde{\Theta}(H_{0,\epsilon})) = 0$, which is equivalent to (5.7) by (3.3), we have

$$\Delta \ln \det \tilde{H}_\epsilon = 0.$$

Hence $\det \tilde{H}_\epsilon = \text{const}$. We normalize $H_{1,\epsilon}$ such that $\det \tilde{H}_\epsilon = 1$.

We first do C^0 -estimates. In order to control H_ϵ , we should estimate $\text{Tr } H_\epsilon$. From [22, p.876], we have

$$(6.4) \quad \Delta \text{Tr } \tilde{H}_\epsilon \leq \text{Tr } \tilde{H}_\epsilon \mid \Lambda\tilde{\Theta}(H_{0,\epsilon}) \mid,$$

which is actually derived from (6.3). We need

Lemma 11. *There is a function $I(\epsilon)$ depending only on ϵ with $I(\epsilon) \geq C\epsilon^{10}$, where C is a constant, such that for any function f on X ,*

$$\|df\|_2^2 \geq I(\epsilon)(\|f\|_4^2 - \|f\|_2^2).$$

Proof. We shall follow the proof in [11]. First, we comment that the lemma is about the estimate of the Sobolev constants. To begin with, because X has volume one and dimension four, following the notation of [18, Lemma 2], for any arbitrary function f over X ,

$$\|df\|_2^2 \geq D(4)C_2(\|f\|_4^2 - \|f\|_2^2).$$

By [18], $D(4)$ is an absolute constant, $C_2 = D(4)C_0^{\frac{1}{2}}$, $2C_1 \geq C_0 \geq C_1$, and C_1 is the constant given by the isoperimetric inequality

$$C_1(\min\{\text{vol}(M_1), \text{vol}(M_2)\})^3 \leq \text{vol}(N)^4$$

of which N runs through all codimension one submanifolds dividing X into two components M_1 and M_2 . Because X is flat and $\text{diam}(X) = \sqrt{2}/\epsilon$, [1, Thm 13] implies

$$C_1 \geq C_4 \left(\int_0^{\text{diam}(X)} r^3 dr \right)^{-5} = C_5 \epsilon^{20}$$

for constants C_4 and C_5 independent of ϵ . Henceforth, $C_0 \geq C_6 \epsilon^{20}$; and for $I(\epsilon)$:

$$I(\epsilon) = \min\{D(4), 1\}C_2 \geq C\epsilon^{10}.$$

□

Then we have

Proposition 12. *For any positive integer l , there is a constant $C(l, r_0)$ such that for any $0 < \epsilon < \frac{3}{16}$,*

$$\mathrm{Tr} \tilde{H}_\epsilon < 2 + C(l, r_0) \epsilon^{\frac{l-11}{2}}.$$

Proof. Let $\tau = \mathrm{Tr} \tilde{H}_\epsilon$, then from (6.4) and (5.12), we have

$$\Delta \tau \leq C_1 \epsilon^{l-1} \tau,$$

where C_1 is a constant depending only on l and r_0 . Hence we have

$$(6.5) \quad \int_X \tau^{2p-1} \Delta \tau \leq C_1 \epsilon^{l-1} \int_X \tau^{2p} \quad \text{for } p \geq 1.$$

Because

$$\int_X \tau^{2p-1} \Delta \tau = (2p-1) p^{-2} \int_X |\nabla \tau^p|^2,$$

then from (6.5),

$$(6.6) \quad \int_X |\nabla \tau^p|^2 p^2 (2p-1)^{-1} C_1 \epsilon^{l-1} \int_X \tau^{2p}.$$

Combined with Lemma 11 we obtain

$$\|\tau\|_{4p}^{2p} \leq (1 + p^2 (2p-1)^{-1} C_2 \epsilon^{l-11}) \|\tau\|_{2p}^{2p} \leq (1 + C_3 \epsilon^{l-11} p) \|\tau\|_{2p}^{2p}.$$

If we set $p = 2^m$, then

$$\|\tau\|_{2^{m+2}}^2 \leq (1 + C_3 \epsilon^{l-11} 2^m)^{\frac{1}{2^m}} \|\tau\|_{2^{m+1}}^2.$$

Iterating the inequality, we obtain

$$(6.7) \quad \|\tau\|_\infty^2 \leq \prod_{m=0}^{\infty} (1 + C_3 \epsilon^{l-11} 2^m)^{\frac{1}{2^m}} \|\tau\|_2^2.$$

It is easy to see that there is a constant C_4 such that

$$(6.8) \quad \prod_{m=0}^{\infty} (1 + C_3 \epsilon^{l-11} 2^m)^{\frac{1}{2^m}} < \exp(C_4 \epsilon^{\frac{l-11}{2}}).$$

It remains to estimate $\|\tau\|_2^2$. First we prove that there exists a point x_0 in X such that $\tau(x_0) = 2$. Such a point x_0 may depend on ϵ . Otherwise $\tau(x) > 2$ for every x in X since we have normalized $H_{1,\epsilon}$ such that $\det H_\epsilon = 1$. Hence two eigenvalues $\lambda_1(x)$ and $\lambda_2(x) = \lambda_1^{-1}(x)$ are not equal for every $x \in X$ and define two smooth functions on X . Thus, there are only two different eigenvalues in V_x up to constant. Normalizing them forms two smooth sections of V . Therefore, V as a complex vector bundle splits into two trivial line bundles. Consequently, $c_1(V) = c_2(V) = 0$, which contradicts to (2.6).

Now we assume that $\tau(x_0) = 2$. Because X is a flat torus, for any $x \in X$, x and x_0 can be joined by a minimal geodesic $\gamma(x)$ (where x is not the cut point of x_0 , the geodesic is unique). Thus, we have

$$\tau(x) \leq \tau(x_0) + \int_{\gamma(x)} \partial_\gamma \tau \leq 2 + \int_{\gamma(x)} |\nabla \tau|.$$

Hence

$$\tau^2 \leq 4 + 4 \int_{\gamma(x)} |\nabla \tau| + \left(\int_{\gamma(x)} |\nabla \tau| \right)^2 \leq 4 + 4 \int_{\gamma(x)} |\nabla \tau| + \int_{\gamma(x)} |\nabla \tau|^2.$$

Using (6.6) for $p = 1$, then,

$$\begin{aligned} \|\tau\|_2^2 &= \int_X \tau^2 \leq 4 + 4 \int_X \int_{\gamma(x)} |\nabla \tau| + \int_X \int_{\gamma(x)} |\nabla \tau|^2 \\ &\leq 4 + 4 \text{diam}(X) \int_X |\nabla \tau| + \text{diam}(X) \int_X |\nabla \tau|^2 \\ &\leq 4 + \frac{C_5}{\epsilon} \epsilon^{\frac{l-3}{2}} \|\tau\|_2 + \frac{C_6}{\epsilon} \epsilon^{l-3} \|\tau\|_2^2 \\ &\leq 4 + C_7 \epsilon^{\frac{l-5}{2}} \|\tau\|_2^2, \end{aligned}$$

and thus,

$$(6.9) \quad \|\tau\|_2^2 \leq \frac{4}{1 - C_7 \epsilon^{\frac{l-5}{2}}}.$$

Now from (6.7), (6.8) and (6.9), we have

$$\|\tau\|_\infty^2 \leq \frac{4 \exp(C_4 \epsilon^{\frac{l-11}{2}})}{1 - C_7 \epsilon^{\frac{l-5}{2}}} \leq 4(1 + C_8 \epsilon^{\frac{l-11}{2}})$$

or

$$\|\tau\|_\infty \leq 2(1 + C_9 \epsilon^{\frac{l-11}{2}}) \leq 2 + C_{10} \epsilon^{\frac{l-11}{2}}.$$

□

Now we are in position to prove C^0 -estimates.

Theorem 13. *For any positive integer l , there is a constant $C(l, r_0)$ such that for any $0 < \epsilon < 1/8$,*

$$\|\check{H}_{1,\epsilon} - I\|_{C^0(\mathcal{U}_\alpha)} < C(l, r_0) \epsilon^{\frac{l-11}{2}}.$$

Proof. By the first equality in (6.1),

$$|\check{H}_{1,\epsilon} - I|^2 = |\check{H}_\epsilon - I|^2 = \text{Tr}(\check{H}_\epsilon - I)^2.$$

On the other hand, $\det \check{H}_\epsilon = \det \tilde{H}_\epsilon = 1$ and $\text{Tr} \check{H}_\epsilon = \text{Tr} \tilde{H}_\epsilon$. Then, by the above proposition, direct calculation proves the theorem. □

Next we estimate $\|\check{H}_{1,\epsilon} - I\|_{C^k(\mathcal{U}_\alpha)}$ for $k \geq 1$. As $\check{H}_{1,\epsilon} = (\check{H}_\epsilon)^t$, we only need to estimate $\|\check{H}_\epsilon - I\|_{C^k(\mathcal{U}_\alpha)}$. Our starting point is (6.2). By this equality, since $H_{1,\epsilon}$ is the HYM metric, formula (3.6) gives

$$0 = \Lambda \tilde{\Theta}(H_{1,\epsilon}) = \Lambda(\bar{\partial}(\partial(B^t(\check{H}_\epsilon)^t \bar{B})(B^t(\check{H}_\epsilon)^t \bar{B}))^{-1})^t,$$

which is equivalent to

$$(6.10) \quad \check{H}_\epsilon B \cdot \Lambda(\bar{\partial}(\partial(B^t(\check{H}_\epsilon)^t \bar{B})(B^t(\check{H}_\epsilon)^t \bar{B}))^{-1})^t \cdot B^{-1} = 0.$$

On the other hand, (5.16) implies $\tilde{H}_{0,\epsilon} = B^t \bar{B}$. Hence formula (3.6) also gives

$$(6.11) \quad \Lambda \tilde{\Theta}(H_{0,\epsilon}) = \Lambda \bar{\partial}(\partial(B^t \bar{B})(B^t \bar{B})^{-1})^t,$$

or

$$(6.12) \quad B \cdot \Lambda \bar{\partial}(\partial(B^t \bar{B})(B^t \bar{B})^{-1})^t \cdot B^{-1} \check{H}_\epsilon = B \cdot \Lambda \tilde{\Theta}(H_{0,\epsilon}) \cdot B^{-1} \check{H}_\epsilon.$$

Subtracting (6.12) from (6.10), expanding the LHS of the resulting equation, and adapting suitably some terms, therefore we obtain

$$\begin{aligned}
(6.13) \quad 0 = & i\Lambda\bar{\partial}\partial\mathcal{H}_\epsilon - i\Lambda\bar{\partial}\mathcal{H}_\epsilon \cdot \check{H}_\epsilon \wedge \partial\mathcal{H}_\epsilon \\
& - i\Lambda\check{H}_\epsilon \cdot \bar{\partial}\log B \cdot \check{H}_\epsilon^{-1} \wedge \partial\mathcal{H}_\epsilon \\
& - i\Lambda\bar{\partial}\mathcal{H}_\epsilon \cdot \check{H}_\epsilon^{-1} \cdot \overline{(\bar{\partial}\log B)^t} \cdot \check{H}_\epsilon \\
& - i\Lambda\partial\mathcal{H}_\epsilon \wedge \bar{\partial}\log B - i\Lambda\overline{(\bar{\partial}\log B)^t} \wedge \bar{\partial}\mathcal{H}_\epsilon \\
& - i\Lambda\mathcal{H}_\epsilon\partial(\bar{\partial}\log B) + i\Lambda\partial(\bar{\partial}\log B) \cdot \mathcal{H}_\epsilon \\
& - i\Lambda\check{H}_\epsilon \cdot \bar{\partial}\log B \cdot \mathfrak{H}_\epsilon \wedge \overline{(\bar{\partial}\log B)^t} \cdot \check{H}_\epsilon \\
& + i\Lambda\check{H}_\epsilon \cdot \mathfrak{H}_\epsilon \cdot \bar{\partial}\log B \wedge \overline{(\bar{\partial}\log B)^t} \cdot \check{H}_\epsilon \\
& - i\Lambda\overline{(\bar{\partial}\log B)^t} \cdot \mathcal{H}_\epsilon \wedge \bar{\partial}\log B \\
& + i\Lambda\overline{(\bar{\partial}\log B)^t} \wedge \bar{\partial}\log B \cdot \mathcal{H}_\epsilon \\
& + i\Lambda B\tilde{\Theta}(H_{0,\epsilon})B^{-1}\check{H}_\epsilon.
\end{aligned}$$

Here for brevity, we have introduced the notations $\mathcal{H}_\epsilon = \check{H}_\epsilon - I$, $\mathfrak{H}_\epsilon = \check{H}_\epsilon^{-1} - I$, and $\bar{\partial}\log B = \bar{\partial}B \cdot B^{-1}$.

We introduce

$$x_{i,\epsilon} = \epsilon^{-1/2}x_i, \quad y_{i,\epsilon} = \epsilon^{1/2}y_i, \quad \text{for } i = 1, 2;$$

and

$$(6.14) \quad z_\epsilon = \epsilon^{-1/2}z, \quad w_\epsilon = \epsilon^{1/2}w.$$

Then (2.1) can be rewritten as

$$(6.15) \quad \omega_\epsilon = dy_{1,\epsilon} \wedge dy_{2,\epsilon} + dx_{1,\epsilon} \wedge dx_{2,\epsilon}.$$

This is the Euclidean metric. We will use ∇_ϵ^k , Δ_ϵ and C_ϵ^k to denote the k -th covariant derivatives, the Laplacian and the C^k -norm with respect to these new coordinates.

The system (6.13) can be rewritten as

$$I_2 = I_1 + I_0,$$

where

$$\begin{aligned}
I_2 = & \frac{\partial^2 \mathcal{H}_\epsilon}{\partial z_\epsilon \partial \bar{z}_\epsilon} + \frac{\partial^2 \mathcal{H}_\epsilon}{\partial w_\epsilon \partial \bar{w}_\epsilon}, \\
I_1 = & \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{z}_\epsilon} \check{H}_\epsilon \frac{\partial \mathcal{H}_\epsilon}{\partial z_\epsilon} + \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{w}_\epsilon} \check{H}_\epsilon \frac{\partial \mathcal{H}_\epsilon}{\partial w_\epsilon} \\
& + \check{H}_\epsilon \frac{\partial \log B}{\partial \bar{z}_\epsilon} \check{H}_\epsilon^{-1} \frac{\partial \mathcal{H}_\epsilon}{\partial z_\epsilon} + \check{H}_\epsilon \frac{\partial \log B}{\partial \bar{w}_\epsilon} \check{H}_\epsilon^{-1} \frac{\partial \mathcal{H}_\epsilon}{\partial w_\epsilon} \\
& + \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{z}_\epsilon} \check{H}_\epsilon^{-1} \overline{\left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t} \check{H}_\epsilon + \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{w}_\epsilon} \check{H}_\epsilon^{-1} \overline{\left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t} \check{H}_\epsilon \\
& - \frac{\partial \mathcal{H}_\epsilon}{\partial z_\epsilon} \frac{\partial \log B}{\partial \bar{z}_\epsilon} - \frac{\partial \mathcal{H}_\epsilon}{\partial w_\epsilon} \frac{\partial \log B}{\partial \bar{w}_\epsilon} \\
& - \overline{\left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t} \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{z}_\epsilon} - \overline{\left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t} \frac{\partial \mathcal{H}_\epsilon}{\partial \bar{w}_\epsilon},
\end{aligned}$$

and

$$\begin{aligned}
I_0 = & -\mathcal{H}_\epsilon \frac{\partial^2 \log B}{\partial z_\epsilon \partial \bar{z}_\epsilon} - \mathcal{H}_\epsilon \frac{\partial^2 \log B}{\partial w_\epsilon \partial \bar{w}_\epsilon} \\
& + \frac{\partial^2 \log B}{\partial z_\epsilon \partial \bar{z}_\epsilon} \mathcal{H}_\epsilon + \frac{\partial^2 \log B}{\partial w_\epsilon \partial \bar{w}_\epsilon} \mathcal{H}_\epsilon \\
& + \check{H}_\epsilon \frac{\partial \log B}{\partial \bar{z}_\epsilon} \check{\mathfrak{H}}_\epsilon \left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t \check{H}_\epsilon + \check{H}_\epsilon \frac{\partial \log B}{\partial \bar{w}_\epsilon} \check{\mathfrak{H}}_\epsilon \left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t \check{H}_\epsilon \\
& - \check{H}_\epsilon \check{\mathfrak{H}}_\epsilon \frac{\partial \log B}{\partial \bar{z}_\epsilon} \left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t \check{H}_\epsilon - \check{H}_\epsilon \check{\mathfrak{H}}_\epsilon \frac{\partial \log B}{\partial \bar{w}_\epsilon} \left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t \check{H}_\epsilon \\
& - \left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t \mathcal{H}_\epsilon \frac{\partial \log B}{\partial \bar{z}_\epsilon} - \left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t \mathcal{H}_\epsilon \frac{\partial \log B}{\partial \bar{w}_\epsilon} \\
& + \left(\frac{\partial \log B}{\partial \bar{z}_\epsilon} \right)^t \frac{\partial \log B}{\partial \bar{z}_\epsilon} \mathcal{H}_\epsilon + \left(\frac{\partial \log B}{\partial \bar{w}_\epsilon} \right)^t \frac{\partial \log B}{\partial \bar{w}_\epsilon} \mathcal{H}_\epsilon \\
& + i \Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1} \check{H}_\epsilon.
\end{aligned}$$

I_1 contains the terms with the first order derivatives of \mathcal{H}_ϵ and I_0 contains the terms with no derivative of \mathcal{H}_ϵ . In I_0 all but the last term have the factor \mathcal{H}_ϵ or $\check{\mathfrak{H}}_\epsilon$, which, by Theorem 13, are very small:

$$(6.16) \quad \|\mathcal{H}_\epsilon\|_{C^0} \leq C(l, r_0) \epsilon^{\frac{l-11}{2}}, \quad \|\check{\mathfrak{H}}_\epsilon\|_{C^0} \leq C(l, r_0) \epsilon^{\frac{l-11}{2}}.$$

We should first deal with the last term.

Combining (3.6) with (2.15) and using (5.16), (5.17) and (5.18), we have

$$\Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1} = \Lambda B A^{-1} \hat{\Theta}(H_{0,\epsilon}) A B^{-1} = \Lambda N \hat{\Theta}(H_{0,\epsilon}) N^{-1} = \Lambda \hat{\Theta}(H_{0,\epsilon}).$$

Hence by (5.11),

$$(6.17) \quad \|\Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1}\|_{C^0} \leq C(l, r_0) \epsilon^{l-1}.$$

Moreover, we recall (5.8):

$$\Lambda \hat{\Theta}(H_{0,\epsilon}) = \psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where ψ is defined in (5.9) and is zero when restricted to \mathcal{U}_0 , \mathcal{U}_j , and $\mathcal{U}_a(r_0)$. Then the first part of Theorem 3 implies

$$\begin{aligned}
(6.18) \quad & \|\Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1}\|_{C^k(\mathcal{U}_\alpha)} \leq \epsilon^{k/2} \|\Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1}\|_{C^k(\mathcal{U}_\alpha)} \\
& \leq C(l, r_0, k) \epsilon^{l-1-k/2}.
\end{aligned}$$

Next we estimate the terms coming from $\bar{\partial} \log B$ and $\partial \bar{\partial} \log B$. The most complicated case is over \mathcal{U}_a . (Note we have shrunk \mathcal{U}_0 in section 5.) Hence we will omit the other cases and only estimate for this case. By (5.16) and (5.14),

$$B = e^{\frac{1}{2}g_a} \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix} A,$$

where κ is defined in (5.15), which can be written as

$$\kappa = \begin{cases} r^{\frac{1}{4}} \left(\frac{1-\rho+\rho \exp(u_\epsilon - \frac{1}{2} \ln r)}{1-\rho+\rho \exp(\frac{1}{2} \ln r - u_\epsilon)} \right)^{\frac{1}{4}} & \text{when } r \in [r_0, 2r_0] \\ e^{\frac{1}{2}u_\epsilon} & \text{when } r \in [0, r_0]. \end{cases}$$

Direct calculation yields

$$\begin{aligned}\frac{\partial \log B}{\partial \bar{z}} &= \frac{1}{2} \frac{\partial g_a}{\partial \bar{z}} I + \frac{\partial \log \kappa}{\partial \bar{z}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \frac{\partial \log B}{\partial \bar{w}} &= \pi i \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}, \\ \frac{\partial^2 \log B}{\partial z \partial \bar{z}} &= \frac{\partial^2 \log \kappa}{\partial z \partial \bar{z}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \frac{\partial^2 \log B}{\partial z \partial \bar{w}} &= \pi i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \frac{\partial^2 \log B}{\partial w \partial \bar{w}} &= 0.\end{aligned}$$

Consequently,

$$(6.19) \quad \left\| \frac{\partial \log B}{\partial \bar{w}_\epsilon} \right\|_{C^0(\mathcal{U})} \leq C\epsilon^{-\frac{1}{2}}, \quad \left\| \frac{\partial \log B}{\partial \bar{w}_\epsilon} \right\|_{C^1(\mathcal{U})} \leq C,$$

and for $k \geq 2$,

$$(6.20) \quad \left\| \frac{\partial \log B}{\partial \bar{w}_\epsilon} \right\|_{C^k(\mathcal{U})} = 0.$$

Moreover, by Theorem 3, if we discuss the C_ϵ^k -norm of $\frac{\partial \log B}{\partial \bar{z}_\epsilon}$ on the domain $\mathcal{U} \setminus \mathcal{U}(r_0)$ and $\mathcal{U}(r_0)$ respectively, we see that

$$(6.21) \quad \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^k(\mathcal{U})} \leq C(r_0, k)\epsilon^{-\frac{k+5}{2}}.$$

By (6.16, 6.17, 6.19, 6.21), we obtain

$$(6.22) \quad \|I_0\|_{C^0} \leq C(l, r_0)\epsilon^{\frac{l-21}{2}}.$$

Having made above preparations, we begin to estimate $\|\nabla_\epsilon^k \mathcal{H}_\epsilon\|_{L^2(\mathcal{U})}$. The approach is standard, so we will not mention basic inequalities such as Young's and Hölder's inequality. We only need to be very careful to deal with ϵ . We assume that \mathcal{H}_ϵ has a compact support in \mathcal{U}_α otherwise we can shrink the open sets \mathcal{U}_α to \mathcal{U}'_α such that they still form an open cover of X and then use cut-off functions. Here we note that we can shrink \mathcal{U}_α to \mathcal{U}'_α by shrinking U_α to U'_α in B and hence the cut-off functions can be taken only defined on B . Thus $|\frac{\partial \chi}{\partial \bar{z}_\epsilon}| = \epsilon^{\frac{1}{2}} |\frac{\partial \chi}{\partial \bar{z}}|$, which is good enough for us to estimate. We will omit the domain \mathcal{U} of integration. We will take C as the generic constant which depends on l, k, r_0 . Remember that \mathcal{H}_ϵ is Hermitian symmetric.

We first do

$$\begin{aligned}\int |\nabla_\epsilon \mathcal{H}_\epsilon|^2 &= - \int \mathcal{H}_\epsilon \triangle_\epsilon \mathcal{H}_\epsilon = - \int \mathcal{H}_\epsilon \cdot (I_1 + I_0) \\ &\leq C \|\mathcal{H}_\epsilon\|_{C^0} \int |\nabla_\epsilon \mathcal{H}_\epsilon|^2 + C \|\mathcal{H}_\epsilon\|_{C^0} \|I_0\|_{C^0}.\end{aligned}$$

When ϵ is small enough, by (6.16), $\|\mathcal{H}_\epsilon\|_{C^0}$ is very small and hence the first term of the RHS can be controlled by the LHS. Combining this with (6.16) and (6.22) gives

$$(6.23) \quad \|\nabla_\epsilon \mathcal{H}_\epsilon\|_{L^2} \leq C(l, r_0)\epsilon^{\frac{l-16}{2}}.$$

Next we estimate

$$\begin{aligned}(6.24) \quad \int |\nabla_\epsilon^2 \mathcal{H}_\epsilon|^2 &\leq \int \triangle_\epsilon \mathcal{H}_\epsilon \triangle_\epsilon \mathcal{H}_\epsilon \leq 2 \int |I_1|^2 + \int |I_0|^2 \\ &\leq C \int |\nabla_\epsilon \mathcal{H}_\epsilon|^4 + C\epsilon^{-5} \int |\nabla_\epsilon \mathcal{H}_\epsilon|^2 + C\epsilon^{l-21}.\end{aligned}$$

We need the following Gagliardo-Nirenberg inequality.

Lemma 14. [20] *Let $f \in L^p(\mathbb{R}^n)$, $D^m f \in L^q(\mathbb{R}^n)$, $l \leq p, q \leq +\infty$. Then for any i ($0 \leq i \leq m$), there exists a constant C such that*

$$\| D^i f \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}^{1-\frac{i}{m}} \| D^m f \|_{L^q(\mathbb{R}^n)}^{\frac{i}{m}},$$

where

$$\frac{1}{r} = \left(1 - \frac{i}{m}\right) \frac{1}{p} + \frac{i}{m} \frac{1}{q}.$$

We use this lemma for the case: $n = 2$, $i = 1$, $r = 4$, $m = 2$, $q = 2$, and $p = +\infty$,

$$\int |\nabla_\epsilon \mathcal{H}_\epsilon|^4 \leq C \| \mathcal{H}_\epsilon \|_{C^0}^2 \int |\nabla_\epsilon^2 \mathcal{H}_\epsilon|^2.$$

Hence, when ϵ is small enough, in (6.24), the first term of the RHS can be controlled by the LHS. Thus, by (6.23), we get

$$(6.25) \quad \| \nabla_\epsilon^2 \mathcal{H}_\epsilon \|_{L^2} \leq C \epsilon^{\frac{l-21}{2}}.$$

Now we use the inductive method to estimate $\| \nabla_\epsilon^k \mathcal{H}_\epsilon \|_{L^2}$. By observation, we find that

$$M_0(k) \triangleq \int |\nabla_\epsilon^k \mathcal{H}_\epsilon|^2 \leq C \sum M_i,$$

where

$$\begin{aligned} M_1 &= \sum \int |\nabla_\epsilon^{i_1} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_2} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_3} \mathcal{H}_\epsilon|^2, \\ &\quad \text{for } i_1, i_2 > 0, i_3 \geq 0, i_1 + i_2 + i_3 = k; \\ M_2 &= \sum \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{k-1-k_1}}^2 \int |\nabla_\epsilon^{i_1} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_2} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_3} \mathcal{H}_\epsilon|^2, \\ &\quad \text{for } i_1, i_2 > 0, i_3 \geq 0, i_1 + i_2 + i_3 = k_1 \leq k-1; \\ M_3 &= \sum \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{k-1-k_1}}^2 \int |\nabla_\epsilon^{k_1} \mathcal{H}_\epsilon|^2, \quad \text{for } 0 \leq k_1 \leq k-1; \\ M_4 &= \sum \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{j_1}}^2 \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{j_2}} \int |\nabla_\epsilon^{i_1} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_2} \mathcal{H}_\epsilon|^2 |\nabla_\epsilon^{i_3} \mathcal{H}_\epsilon|^2, \\ &\quad \text{for } i_1, i_2 > 0, i_3 \geq 0, i_1 + i_2 + i_3 = k_1 \leq k-2, j_1 + j_2 = k - k_1 - 2; \\ M_5 &= \sum \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{j_1}}^2 \left\| \frac{\partial \log B}{\partial \bar{z}_\epsilon} \right\|_{C_\epsilon^{j_2}} \int |\nabla_\epsilon^{k_1} \mathcal{H}_\epsilon|^2, \\ &\quad \text{for } 0 \leq k_1 \leq k-2, j_1 + j_2 = k - k_1 - 2; \\ M_6 &= \sum \left\| \Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1} \right\|_{C_\epsilon^{k-2-k_1}}^2 \int |\nabla_\epsilon^{k_1} \mathcal{H}_\epsilon|^2, \quad \text{for } 0 < k_1 \leq k-2; \\ M_7 &= \left\| \Lambda B \tilde{\Theta}(H_{0,\epsilon}) B^{-1} \right\|_{C_\epsilon^{k-2}}^2. \end{aligned}$$

We first deal with M_1 . When $i_1 < k_1$, Lemma 14 implies

$$\left(\int |\nabla_\epsilon^{i_1} \mathcal{H}_\epsilon|^{\frac{2k_1}{i_1}} \right)^{\frac{i_1}{2k_1}} \leq C \| \mathcal{H}_\epsilon \|_{C^0}^{1-\frac{i_1}{k_1}} \left(\int |\nabla_\epsilon^{k_1} \mathcal{H}_\epsilon|^2 \right)^{\frac{i_1}{2k_1}}.$$

Then the Hölder inequality and this inequality implies that the summands of M_1 for $i_3 > 0$ are less than

$$\begin{aligned} & \left(\int |\nabla_\epsilon^{i_1} \mathcal{H}_\epsilon|^{\frac{2k}{i_1}} \right)^{\frac{i_1}{k}} \left(\int |\nabla_\epsilon^{i_2} \mathcal{H}_\epsilon|^{\frac{2k}{i_2}} \right)^{\frac{i_2}{k}} \left(\int |\nabla_\epsilon^{i_3} \mathcal{H}_\epsilon|^{\frac{2k}{i_3}} \right)^{\frac{i_3}{k}} \\ & \leq C \|\mathcal{H}_\epsilon\|_{C^0}^4 \int |\nabla_\epsilon^k \mathcal{H}_\epsilon|^2; \end{aligned}$$

as the same reason, the summands of M_1 for $i_3 = 0$ are less than

$$C \|\mathcal{H}_\epsilon\|_{C^0}^2 \int |\nabla_\epsilon^k \mathcal{H}_\epsilon|^2.$$

Hence when ϵ is small enough, M_1 can be controlled by $M_0(k)$.

As the above discussion, we see that M_2 and M_4 can be controlled by M_3 and M_5 respectively. Thus we get

$$M_0(k) \leq C(M_3 + M_5 + M_6 + M_7).$$

If we let $M_0(k_1) \leq C\epsilon^{f(k_1)}$ for any $k_1 \leq k-1$, then by (6.21) and (6.16),

$$\begin{aligned} \text{the summand of } M_3 & \leq \begin{cases} C\epsilon^{-k+k_1-4+f(k_1)}, & \text{for } 0 < k_1 \leq k-1, \\ C\epsilon^{l-k-15}, & \text{for } k_1 = 0; \end{cases} \\ \text{the summand of } M_5 & \leq \begin{cases} C\epsilon^{-k+k_1-8+f(k_1)}, & \text{for } 0 < k_1 \leq k-2, \\ C\epsilon^{l-k-19}, & \text{for } k_1 = 0; \end{cases} \\ \text{the summand of } M_6 & \leq C\epsilon^{2l-k+k_1+f(k_1)}, \quad \text{for } 0 < k_1 \leq k-2; \\ M_7 & \leq C\epsilon^{2l-k-2}. \end{aligned}$$

Clearly, M_6 and M_7 can be controlled by M_3 . Hence $M_0 \leq C(M_3 + M_5)$. Moreover, (6.23) and (6.25) imply $f(1) = l-16$ and $f(2) = l-21$. If we let $f(k_1) = l-11-5k_1$ for any integer $1 \leq k_1 \leq k-1$, then we see that

$$M_0(k) \leq C\epsilon^{-5+f(k-1)} = C\epsilon^{l-11-5k}.$$

Therefore by the inductive method, $f(k) = l-11-5k$, i.e.,

$$\int |\nabla_\epsilon^k \mathcal{H}_\epsilon|^2 \leq C\epsilon^{l-11-5k}.$$

By the Sobolev inequality, we get

$$\|\mathcal{H}_\epsilon\|_{C_\epsilon^{k,\delta}(\mathcal{U})} \leq C\epsilon^{\frac{l-15-5k}{2}},$$

which is transferred, by (6.14), to

$$\|\mathcal{H}_\epsilon\|_{C^{k,\delta}(\mathcal{U})} \leq C\epsilon^{\frac{l-15-6k}{2}}.$$

Thus we obtain, since $\mathcal{H}_\epsilon = \check{H}_{1,\epsilon} - I$,

Theorem 15. *For any positive integers l and k so that $l > 6k + 15$, there is a constant $C = C(l, k, \mathcal{U}'_\alpha)$ depending on l , k and an open cover $\{\mathcal{U}'_\alpha\}$ of X which is smaller than $\{\mathcal{U}_\alpha\}$ such that for any positive ϵ small enough,*

$$\|\check{H}_{1,\epsilon} - I\|_{C^k(\mathcal{U}'_\alpha)} \leq C\epsilon^{\frac{l-15-6k}{2}}.$$

Now we can prove Theorem 2.

Proof. For the unitary frame $(\check{\mu}_1, \check{\mu}_2)$ associated to $H_{0,\epsilon}$, the resulting matrix representations of $(H_{0,\epsilon})^{-1}H_{1,\epsilon}$ are $\check{H}_{1,\epsilon}$. Moreover, we can replace $\frac{l-15-6k}{2}$ by l' . Thus the above theorem implies Theorem 2. \square

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SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA
E-mail address: majxfu@fudan.edu.cn